Poincaré invariance for continuous-time histories

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Abstract

We show that the relativistic analogue of the two types of time translation in a non-relativistic history theory is the existence of two distinct *Poincaré groups*. The 'internal' Poincaré group is analogous to the one that arises in the standard canonical quantisation scheme; the 'external' Poincaré group is similar to the group that arises in a *Lagrangian* description of the standard theory. In particular, it *performs explicit changes of the spacetime foliation* that is implicitly assumed in standard canonical field theory.

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1 Introduction

The generalisation of continuous-time history theory to include relativistic quantum fields raises some subtle issues that tend to be hidden in the normal canonical treatment of a quantum field.

The standard canonical quantisation of a relativistic field requires the choice of a Lorentzian foliation on the background spacetime: the Hamiltonian is then defined with respect to this foliation. There exist many unitarily inequivalent representations of the canonical commutation relations for this quantum field theory: the physically appropriate one is chosen by requiring that the Hamiltonian exists as a well-defined self-adjoint operator. In this sense—like the Hamiltonian itself—the physically appropriate representation is foliation-dependent. Relativistic covariance is then implemented by seeking a representation of the Poincaré group on the resulting Hilbert space. However, the Poincaré group thus constructed does not explicitly perform a *change of the foliation*.

The HPO continuous-time histories approach to quantum theory [1, 2, 3, 4] is particularly suited to deal with systems that have a non-trivial temporal structure, and therefore it should be able to provide a significant clarification of this point.

Specifically, we will show that the relativistic analogue of the two types of time translation that arise in a non-relativistic history theory is the existence of two distinct *Poincaré groups*. The 'internal' Poincaré group is analogous to the one that arises in the standard canonical quantisation scheme as sketched above.

However, the 'external' one is a novel object: it is similar to the group that arises in the *Lagrangian* description of the field theory. In particular, it explicitly performs *changes* of the foliation. This arises from the striking property that HPO theories admit two distinct types of time transformation, each representing a distinct quality of time [1]. The first corresponds to time considered purely as a kinematical parameter of a physical system, with respect to which a history is defined as a succession of possible events. It is strongly connected with the temporal-logical structure of the theory and is related to the view of time as a parameter that determines the ordering of events. The second corresponds to the dynamical evolution generated by the Hamiltonian. For a detailed presentation of the HPO continuous-time programme see [1].

As we shall see, one of the important results of the formalism as applied to a field theory is that, even though the representations of the history algebra are foliation dependent, the physical quantities (probabilities) are *not*.

In section 2, we shall give a brief description of the underlying concepts of the continuous-histories programme: this is necessary for establishing the framework of the ensuing work.

In section 3, we present the histories version of a classical scalar field theory: in particular, we show how two Poincaré groups arise as an analogue of the two types of time transformation in the non-relativistic history theory.

The free quantum scalar field theory is presented in section 4. We show that due to the histories temporal structure previously introduced in [1], manifest Poincaré invariance is possible. Specifically, we show how different representations of the history algebra—corresponding to different choices of foliation—are realised on the *same* Fock space (notwithstanding the fact that the different representations are unitarily inequivalent), and we show that they are related in a certain way with Poincaré transformations.

2 The History Projection Operator Approach

The History Projection Operator (the, so-called, 'HPO' approach) theory was a development [2] (emphasizing quantum *temporal* logic) of the consistent-histories approach to quantum theory inaugurated by Griffiths, Omnés, Gell-Mann and Hartle [5]. However, the novel temporal structure introduced in [1] led to a departure from the original ideas on decoherence. In particular, in our approach, emphasis is placed on the distinction between (i) the temporal logic structure of the theory; and (ii) the dynamics [3].

In consistent-histories theory, a history is defined as a sequence of time-ordered propositions about properties of a physical system, each of which can be represented, as usual, by a projection operator. In normal quantum theory, it is not possible to assign a probability measure to the set of all histories. However, when a certain 'decoherence condition' is satisfied by a set of histories, the elements of this set *can* be given probabilities.

The probability information of the theory is encoded in the decoherence functional: a complex function of pairs of histories which—in the original approach of Griffiths *et al*—can be written as

$$d(\alpha, \beta) = \mathrm{tr}(\tilde{C}_\alpha^\dagger \rho \tilde{C}_\beta) \tag{2.1}$$

where ρ is the initial density-matrix, and where the *class operator* $\tilde{C}_\alpha$ is defined in terms of the standard Schrödinger-picture projection operators α_{t_i} as

$$\tilde{C}_\alpha := U(t_0, t_1)\alpha_{t_1} U(t_1, t_2)\alpha_{t_2} \ldots U(t_{n-1}, t_n)\alpha_{t_n} U(t_n, t_0) \tag{2.2}$$

where $U(t, t') = e^{-i(t-t')H/\hbar}$ is the unitary time-evolution operator from time t to t'. Each projection operator α_{t_i} represents a proposition about the system at time t_i, and the class operator $\tilde{C}_\alpha$ represents the composite history proposition "α_{t_1} is true at time t_1, and then α_{t_2} is true at time t_2, and then $\ldots$, and then α_{t_n} is true at time t_n".

Isham and Linden developed the consistent-histories formalism further, concentrating on its *temporal* quantum logic structure [2]. They showed that propositions about the histories of a system could be represented by *projection operators* on a new, 'history' Hilbert space. In particular, the history proposition "α_{t_1} is true at time t_1, and then α_{t_2} is true at time t_2, and then $\ldots$, and then α_{t_n} is true at time t_n" is represented by the *tensor product* $\alpha_{t_1} \otimes \alpha_{t_2} \otimes \cdots \otimes \alpha_{t_n}$ which, unlike $\tilde{C}_\alpha$, *is* a genuine projection operator, that is defined on the tensor product of copies of the standard Hilbert space $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n}$. Hence the 'History Projection Operator' formalism extends to multiple times, the quantum logic of single-time quantum theory.

The history space. An important way of understanding the history Hilbert space $\mathcal{F}$ is in terms of the representations of the 'history group'—in elementary systems this is the history analogue of the canonical group [2]. For example, for the simple case of a point particle moving on a line, the Lie algebra of the history group for a *continuous* time parameter t is described by the history commutation relations

$$[x_t, x_{t'}] = 0 \tag{2.3}$$
$$[p_t, p_{t'}] = 0 \tag{2.4}$$
$$[x_t, p_{t'}] = i\hbar\delta(t - t') \tag{2.5}$$

where $-\infty \leq t, t' \leq \infty$. It is important to note that these operators are in the *Schrödinger* picture, and that the history algebra is invariant under translations of the time index of these operators.

The choice of the Dirac delta-function in the right hand side of Eq. (2.5) is associated with the requirement that time be treated as a continuous variable.

One important consequence is the fact that the observables cannot be defined at sharp moments of time but rather appear naturally as *time-averaged*.

A unique representation of this algebra can be found by requiring the existence of an operator analogue of a time-averaged Hamiltonian $H = \int_{-\infty}^{\infty} dt\, H_t$, where H_t is the standard Hamiltonian defined at a moment of time t [6].

The Action and Liouville operators. One of the original problems in the development of the HPO theory was the lack of a clear notion of time evolution, in the sense that, there was no natural way to express the time translations from one time slot—that refers to one copy of the Hilbert space $\mathcal{H}_t$—to another one, that refers to another copy $\mathcal{H}_{t'}$. The situation changed with the introduction of the *'action' operator* S.

Indeed, the crucial step for constructing the temporal structure of the theory was the definition in [1] of the action operator S—a quantum analogue of the Hamilton-Jacobi functional [7], written as

$$S_\kappa := \int_{-\infty}^{+\infty} dt\ (p_t \dot{x}_t - \kappa(t) H_t), \tag{2.6}$$

where $\kappa(t)$ is an appropriate test function.

The first term of the action operator S_κ Eq. (2.6) is identical to the kinematical part of the classical phase space action functional. This 'Liouville' operator is formally written as

$$V := \int_{-\infty}^{\infty} dt\, (p_t \dot{x}_t) \tag{2.7}$$

so that

$$S_\kappa = V - H_\kappa. \tag{2.8}$$

2.1 The temporal structure

A fundamental property of the HPO form of history theory is that the Liouville operator V and the Hamiltonian operator H_κ generate two distinct types of time transformation. The Liouville operator V relates the Schrödinger-picture operators associated with different time-t labels, whereas H_t is associated with internal dynamical changes at the fixed time t (with an analogous statement for the smeared operator H_κ). The action operator S_κ is thus the generator of both types of time translation [1].

More precisely, it was shown that there exist *two* distinct types of time transformation. One—generated by the Liouville operator V—refers to time as it appears in temporal logic, and it is related to t-label in Eqs. (2.3–2.5). The other—generated by the Hamiltonian—refers to time as it appears in the implementation of dynamical laws, and it is related to the label s in the 'history Heisenberg picture' operator, that is hence defined in accord to the novel conceptual issues introduced with the 'two modes of time'

$$x_t(s) := e^{isH/\hbar}\, x_t\, e^{-isH/\hbar}. \tag{2.9}$$

where H is defined to be H_κ with κ set equal to 1.

We will use the notation $x_f(s)$ for these history Heisenberg-picture operators smeared with respect to the time label t, and we notice from Eq. (2.9) that these quantities behave like standard Heisenberg-picture operators with a time parameter s.

For any specific physical system these two transformations are intertwined with the aid of the action operator S as

$$e^{i\tau S/\hbar}\, x_f(s)\, e^{-i\tau S/\hbar} = x_{f_\tau}(s+\tau) \tag{2.10}$$

where $f_\tau(t) := f(t+\tau)$, and where S means S_κ with $\kappa = 1$.

Classical histories theory

The continuous-time histories description has a natural analogue for classical histories [3]. In this scheme, the basic mathematical entity is the space $\Pi = C(\mathbb{R}, \Gamma)$ of differentiable paths taking their value in the manifold Γ of classical states. Hence an element of Π is a smooth path $\gamma : \mathbb{R} \to \Gamma$. In effect, we associate a copy of the classical state space with each moment of time, and employ differentiable sections of the ensuing bundle over $\mathbb{R}$.

The key idea in this approach to classical histories is contained in the symplectic structure on this space of temporal paths Π. For example, for a particle moving in one dimension (with configuration coordinate x and momentum coordinate p), the history space Π is equipped with a symplectic form

$$\omega = \int dt\, dp_t \wedge dx_t \tag{2.11}$$

which generates the history Poisson brackets

$$\{x_t, x_{t'}\} = 0 \quad (2.12)$$

$$\{p_t, p_{t'}\} = 0 \quad (2.13)$$

$$\{x_t, p_{t'}\} = \delta(t - t') \quad (2.14)$$

In general, given a function f on Γ we can define an associated family $t \mapsto F_t$ of functions on Π as

$$F_t(\gamma) := f(\gamma(t)). \quad (2.15)$$

In this way, all transformations implemented through the Poisson bracket in the normal canonical theory, correspond to transformations in the history theory that *preserve the time label* t. Indeed, for two families of functions $t \mapsto F_t$ and $t \mapsto G_t$ defined through (2.15) we have

$$\{F_t, G_{t'}\} = L_t \delta(t, t'), \quad (2.16)$$

where L_t corresponds to the function l on Γ

$$l = \{f, g\}_{\Gamma}. \quad (2.17)$$

In this way all relevant structures of the canonical theory can be naturally transferred to the histories framework [3].

The Liouville, Hamilton and action functionals on Π are defined respectively as

$$V(\gamma) := \int_{-\infty}^{\infty} dt\, [p_t \dot{x}_t](\gamma) \quad (2.18)$$

$$H(\gamma) := \int_{-\infty}^{\infty} dt\, [H_t(p_t, x_t)](\gamma) \quad (2.19)$$

$$S(\gamma) := V(\gamma) - H(\gamma) \quad (2.20)$$

where $\dot{x}_t(\gamma) = (\partial x_t / \partial t)(\gamma)$ is the velocity at the time point t of the path γ. These definitions are crucial for the dynamics of the theory. In particular, V and H are the classical analogues of the generators of the two types of time transformation in the history quantum theory [1].

The crucial result of classical histories theory is that one may deduce the equations of motion in the following way [1]: a classical history γ_{cl} is the

realised path of the system—*i.e.* a solution of the equations of motion of the system—if it satisfies the equations

$$\{x_t, V\}(\gamma_{cl}) = \{x_t, H\}(\gamma_{cl}) \tag{2.21}$$

$$\{p_t, V\}(\gamma_{cl}) = \{p_t, H\}(\gamma_{cl}) \tag{2.22}$$

where γ_{cl} is the path $t \mapsto (x_t(\gamma_{cl}), p_t(\gamma_{cl}))$, and $x_t(\gamma_{cl})$ is the position coordinate of the realised path γ_{cl} at the time point t.

The above equations (2.21–2.22) are the history equivalent of the canonical equations of motion. In particular, the symplectic transformation generated by the history action functional $S(\gamma)$ leaves invariant the paths that are classical solutions of the system:

$$\{x_t, S\}(\gamma_{cl}) = 0 \tag{2.23}$$

$$\{p_t, S\}(\gamma_{cl}) = 0. \tag{2.24}$$

More generally, any function F on Π satisfies the equation

$$\{F, S\}(\gamma_{cl}) = 0. \tag{2.25}$$

This is the way in which equations of motion appear in the classical history theory. Notice that the role of the action as the generator of time transformations emerges naturally in this classical case. Furthermore, the condition (2.25) above emphasises the role of the Hamiltonian and Liouville functionals in histories theory as generators of different types of time transformation. It also clarifies the new temporal structure that arises in history theory when compared with the standard classical theory.

This result is of particular importance in the case of parameterised systems, where the notion of time is recovered *after the phase space reduction* [3].

3 Classical Scalar Field Theory

3.1 Background

Standard canonical treatment

In the Hamiltonian description of a free scalar field ϕ with mass $\tilde{m}$ on Minkowski spacetime, the first step is to choose a spacelike foliation, which

can be specified by its normal—a unit time-like vector n^μ. We shall take the signature of the Minkowski metric $\eta^{\mu\nu}$ to be $(+,-,-,-)$.

The first step is to select a specific foliation, and to choose a reference leaf $\Sigma \simeq \mathbb{R}^3$ that is characterised by $t=0$, where t is the natural time label associated with the foliation.

The corresponding configuration space is the space $C^\infty(\Sigma)$ of all smooth scalar functions $\phi(x)$ on Σ, while the phase space Γ is its cotangent bundle $T^*C^\infty(\Sigma)$ defined in an appropriate way[1]. The key point about this structure is that the state space of fields is equipped with the Poisson brackets

$$
\begin{aligned}
\{\phi(\underline{x}), \phi(\underline{x}')\} &= 0 && (3.1)\\
\{\pi(\underline{x}), \pi(\underline{x}')\} &= 0 && (3.2)\\
\{\phi(\underline{x}), \pi(\underline{x}')\} &= \delta(\underline{x}-\underline{x}'). && (3.3)
\end{aligned}
$$

Poincaré group symmetry. The relativistic scalar field theory is covariant under the action of the Poincaré group [8]. For a free massive scalar field, the generators of time-translations P^0, space translations P^i, spatial rotations J^i and Lorentz boosts K^i are respectively [2]

$$
\begin{aligned}
H &= P^0 = \frac{1}{2}\int d^3\underline{x}\,[\pi^2 + \partial_i\phi\partial_i\phi + \tilde{m}^2\phi^2] && (3.4)\\
P^i &= \int d^3\underline{x}\,\pi\partial^i\phi && (3.5)\\
J^i &= \frac{1}{2}\epsilon^{ijk}\int d^3\underline{x}\,\pi x_j\partial_k\phi && (3.6)\\
K^i &= M^{0i} = \int d^3\underline{x}\,[t\pi\partial^i\phi - x^i\frac{1}{2}(\pi^2 + \partial_j\phi\partial_j\phi + \tilde{m}^2\phi^2)] && (3.7)
\end{aligned}
$$

where we note that the sub/superscripts i,j,k refer to coordinates in the surface Σ that is spatial with respect to the chosen foliation vector n. Similarly, the integrals above are all defined over Σ.

[1] To make these statements mathematically rigorous it would be necessary to invoke the differential geometry of infinite-dimensional spaces like $C^\infty(\Sigma)$. However, we do not need to become involved in such complexities here: for our purposes it suffices to postulate the basic Poisson algebra relations (3.11–3.13) that follow.

[2] They are obtained by the use of Noether's theorem on the Lagrangian theory, and a Legendre transform.

If we define the partial differential operator

$$(\Gamma f)(\underline{x}) := \Big[(\eta^{\mu\nu} - n^\mu n^\nu)\partial_\mu\partial_\nu + \tilde{m}^2\Big] f(\underline{x}), \tag{3.8}$$

we can write the convenient expressions for the Hamiltonian and the boosts generator as

$$H = \frac{1}{2}\int d^3\underline{x}\,[\pi^2 + \phi\Gamma\phi] \tag{3.9}$$

$$K^i = \int d^3\underline{x}\,[t\pi\partial^i\phi - \frac{1}{2}x^i(\pi^2 + \phi\Gamma\phi)]. \tag{3.10}$$

3.2 Histories description for the classical scalar field

In the histories formalism of a scalar field, the space of phase-space histories[3] Π is an appropriate subset of the continuous Cartesian product $\times_t \Gamma_t$ of copies of the standard state space Γ, each labeled by the time parameter t. The choice of Γ depends on the choice of a foliation vector n^μ, hence the space of histories also has an implicit dependence on n^μ and should therefore be written as ${}^n\Pi$. Furthermore, we write $\Sigma_t = (n,t)$, the space-like surface Σ defined with respect to its normal vector n, and labeled by the parameter t.

To be more precise, for each space-like surface Σ_t we consider the state space $\Gamma_t = T^*C^\infty(\Sigma_t)$. Then we define the fiber bundle with basis $\mathbb{R}$ and fiber Γ_t, at each $t \in \mathbb{R}$. Histories are defined as the cross-sections of the ensuing bundle, and the history space ${}^n\Pi$ is the space of all smooth cross-sections of this bundle.

The Poisson algebra relations of the history theory are

$$\{\ \phi(X)\,, \phi(X')\ \} = 0 \tag{3.11}$$

$$\{\ \pi(X)\,, \pi(X')\ \} = 0 \tag{3.12}$$

$$\{\ \phi(X)\,, \pi(X')\ \} = \delta^4(X - X') \tag{3.13}$$

where X and X' are space-time points. The field $\phi(X)$ and its conjugate momentum $\pi(X')$ are implicitly defined with respect to the foliation vector n^μ.

[3] One may write a history version of the Lagrangian treatment, however this description is not relevant to the immediate aims of this work.

The definitions of the action S, Liouville V and 'Hamiltonian' H functionals are

$$S := V - \frac{1}{2}\int d^4 X \,\{\pi^2(X) + \phi(X)\ {}^n\Gamma\, \phi(X)\} \tag{3.14}$$

$$V := \int d^4 X\, \pi(X) n^\mu \partial_\mu\, \phi(X) \tag{3.15}$$

$$H := \frac{1}{2}\int d^4 X\ \{\pi^2(X) + \phi(X)\ {}^n\Gamma\ \phi(X)\} \tag{3.16}$$

respectively, where again there is an implicit n label on these three quantities; and where Γ is the differential operator

$$\Gamma(X) := \left[(\eta^{\mu\nu} - n^\mu n^\nu)\partial_\mu\partial_\nu + \tilde{m}^2\right] \tag{3.17}$$

introduced above.

As we explained earlier, the variation of $S[\,\gamma\,]$ leaves invariant the paths γ_{cl} that are classical solutions of the system:

$$\{\phi(X), S\}(\gamma_{cl}) = 0 \tag{3.18}$$

$$\{\pi(X), S\}(\gamma_{cl}) = 0 \tag{3.19}$$

As we shall now see, H is the generator to the time averaged internal Poincaré group.

3.3 Poincaré symmetry

The Poincaré group is the group of isometries of the Minkowski metric. Hence, any field theory in Minkowski space-time needs to be covariant under the action of the Poincaré group. As we shall now see, in a history theory—because of its augmented temporal structure—the associated group theory leads to a particular interesting result: namely, there are *two* distinct Poincaré groups that act on the history space.

3.3.1 The internal Poincaré group

One significant feature of histories theory is that it gives a representation of the temporal logic of the system that is *independent* of the dynamics involved. Hence, propositions about the state of the system at different times

are represented by appropriate subsets of the space of paths. In the context of symmetries, however, the temporal logic structure entails the following.

For each copy Γ_t of the standard state space, there exists a Poincarè group symmetry of the type one would expect in a canonical treatment of relativistic field theory. On the other hand, in the history theory the state space is heuristically the Cartesian product of such copies, and all physical quantities in the standard treatment now appear as naturally time-averaged [1]. Hence one may write time-averaged generators of the *internal* Poincaré groups, in a covariant-like notation as

$$H = \frac{1}{2}\int d^4X\,\{\pi(X)^2 + \phi(X)\,{}^n\Gamma\,\phi(X)\} \tag{3.20}$$

$$P(m) = m_\mu \int d^4X\,\pi(X)\,\partial^\mu\,\phi(X) \tag{3.21}$$

$$J(m) = \frac{1}{2} n_\mu m_\nu \epsilon^{\mu\nu\rho\sigma} \int d^4X\,\pi(X)\,X_\rho\,\partial_\sigma\,\phi(X) \tag{3.22}$$

$$K(m) = m_\mu \int d^4X\,\{n\cdot X\,\pi(X)\partial^\mu\phi(X) - \frac{1}{2}X^\mu[\pi(X)^2 + \phi(X)\,{}^n\Gamma\phi(X)\} \tag{3.23}$$

where m^μ is an 'n-spacelike' vector, *i.e.* one such that $n\cdot m := n^\mu m^\nu \eta_{\mu\nu} = 0$.

Of special interest are the groups of canonical transformations generated by the Hamiltonian generator H and the boosts generator K. Note that a space-time point X can be associated with the pair $(t, \underline{x}) \in \mathbb{R}\times\mathbb{R}^3$, as $X = tn + x_n$, where the three-vector $\underline{x}$ has been associated with a corresponding n-spatial four-vector x_n (i.e., $n\cdot x_n = 0$); note that $t = n\cdot X$. Then we define the classical analogue of the Heisenberg picture fields as

$$\phi(X) \;\xrightarrow{H}\; \phi(X, s) \tag{3.24}$$

or

$$\phi(t, \underline{x}) \;\xrightarrow{H}\; \phi(t, \underline{x}, s) := \cos({}^n\Gamma^{\frac{1}{2}} s)\phi(X) + \frac{1}{{}^n\Gamma^{\frac{1}{2}}}\sin({}^n\Gamma^{\frac{1}{2}} s)\pi(X), \tag{3.25}$$

where $\phi(X) := \phi(t, \underline{x})$ and $\phi(X, s) := \phi(t\,, \underline{x}\,, s)$. The square-root operator ${}^n\Gamma^{\frac{1}{2}}$, and functions thereof, can be defined rigorously using the spectral theory of the self-adjoint, partial differential operator ${}^n\Gamma$ on the Hilbert space $L^2(\mathbb{R}^4, d^4X)$.

Notice also that the time label t is not affected by this transformation since $[n \cdot \partial, {}^{n}\mathrm{T}] = 0$. For a fixed value of time t, the field $\phi(t\,,\underline{x}\,,s)$ is the 'Heisenberg-picture' field of the standard canonical treatment.

The action of boost transformations is best shown upon objects $\phi(X,s) = \phi(t\,,\underline{x}\,,s)$ as

$$\phi(t,\underline{x},s) \to \phi(t,\underline{x}',s'), \tag{3.26}$$

where $(\underline{x}',s')$ and $(\underline{x},s)$ are related by the Lorentz boost parametrised by m^{μ} as

$$\begin{aligned} s' &= \cosh|m|s + \frac{\sinh|m|}{|m|}x^i m_i \\ x^{i'} &= (\delta_{ij} - \frac{m^i m_j}{|m|^2})x^j + \frac{m^i m_j}{|m|^2}\cosh|m|x^j + \frac{\sinh|m|}{|m|}m^i s \end{aligned} \tag{3.27}$$

where, as above, x_i is the spatial part of X with respect to n, so that $X = tn + x_n$ and $i = 1,2,3$.

Hence, for each copy of the standard classical state space, there exists an 'internal' Poincarè group that acts on the copy of standard canonical field theory that is labeled by the same t-time label.

3.3.2 The external Poincaré group

For each fixed n, there also exists an '*external*' Poincaré group with generators

$$\tilde{P}^{\mu} = \int d^4X\, \pi(X)\partial^{\mu}\phi(X) \tag{3.28}$$

$$\tilde{M}^{\mu\nu} = \int d^4X\, \pi(X)(X^{\mu}\partial^{\nu} - X^{\nu}\partial^{\mu})\phi(X) \tag{3.29}$$

where $\mu,\nu = 0,1,2,3$ and $\tilde{P}^{\mu}$ generate spacetime translations. The n-spatial parts of the tensor $\tilde{M}^{\mu\nu}$ generate spatial rotations; the time parts generates boosts.

The space translations and rotations are identical to those of the *internal* Poincaré group. However the time translation and the boosts differ. Indeed, under $V := \tilde{P}^0$ we have

$$\phi(t,\underline{x}) \xrightarrow{V} \phi(t+\tau,\underline{x}) \tag{3.30}$$

$$\pi(t,\underline{x}) \xrightarrow{V} \pi(t+\tau,\underline{x}). \tag{3.31}$$

where τ is the time translation generated by V. Thus, what we have shown here is that the time-translation generator for the 'external' Poincaré group is the Liouville functional V.On the other hand, the boost generator $\tilde{K}^i = \tilde{M}^{0i}$ generates Lorentz transformations of the type

$$\phi(X) \to \phi(\Lambda X) \tag{3.32}$$

$$\pi(X) \to \pi(\Lambda X) \tag{3.33}$$

where for future convenience we write as Λ the element of the Lorentz group obtained by exponentiation of the boost parameterised by m^i.

Furthermore, under the action of this external group, the generators of the *internal* Poincaré group transform as follows

$${}^{n}H \quad \xrightarrow{\tilde{K}} \quad {}^{\Lambda n}H \tag{3.34}$$

$${}^{n}K(m) \quad \xrightarrow{\tilde{K}} \quad {}^{\Lambda n}K(\Lambda m). \tag{3.35}$$

where we have now attached the explicit n labels that were implicit in our previous notation for these quantities. The action functional transforms in the same way

$${}^{n}S \quad \to \quad {}^{\Lambda n}S \tag{3.36}$$

Note that the action of the two groups coincides on classical solutions γ_{cl}:

$$\{\phi(X), K(m)\}(\gamma_{cl}) \quad = \quad \{\phi(X), \tilde{K}(m)\}(\gamma_{cl}) \tag{3.37}$$

$$\{\pi(X), K(m)\}(\gamma_{cl}) \quad = \quad \{\pi(X), \tilde{K}(m)\}(\gamma_{cl}) \tag{3.38}$$

We must emphasise again that the definition of Π depends on the foliation vector. Hence, so will the action of the Poincaré group. Here we deal with the scalar field, for which this dependence is not explicit. However, this dependence, and analogue of the Poincaré group action is a major feature in systems where there is an explicit foliation dependence. For example, this is the case of general relativity which is discussed in [9].

4 Histories Quantum Scalar Field Theory

4.1 Background

Canonical quantum field theory

Canonical quantisation proceeds by looking for a representation of the *canonical commutation relations*

$$\begin{aligned} [\, \hat{\phi}(\underline{x}) \,, \hat{\phi}(\underline{x}') \,] &= 0 && (4.1) \\ [\, \hat{\pi}(\underline{x}) \,, \hat{\pi}(\underline{x}') \,] &= 0 && (4.2) \\ [\, \hat{\phi}(\underline{x}) \,, \hat{\pi}(\underline{x}') \,] &= i\hbar\delta^3(\underline{x} - \underline{x}') && (4.3) \end{aligned}$$

on a Hilbert space which, in practice, is selected by requiring the existence of the Hamiltonian as a genuine (essentially) self-adjoint operator.

For a free field, such a representation can be found on the Fock space $\mathcal{F} = \exp L^2(\mathbb{R}^3,\, d^3\underline{x})$ on which the fields can be written in terms of the creation and annihilation operators b and $b^\dagger$ that define $\mathcal{F}$

$$\begin{aligned} \hat{\phi}(\underline{x}) &= \frac{1}{\sqrt{2}}{}^n\Gamma^{-1/4}(\hat{b}(\underline{x}) + \hat{b}^\dagger(\underline{x})) && (4.4) \\ \hat{\pi}(\underline{x}) &= \frac{1}{\sqrt{2}}{}^n\Gamma^{1/4}(\hat{b}(\underline{x}) - \hat{b}^\dagger(\underline{x})) && (4.5) \end{aligned}$$

where

$$[b(\underline{x}), b^\dagger(\underline{x}')] = \delta^3(\underline{x} - \underline{x}'). \qquad (4.6)$$

The (normal-ordered) Hamiltonian then reads[4]

$$\hat{H} = \int d^3\,\underline{x}\; \hat{b}^\dagger(\underline{x})\, {}^n\Gamma\, \hat{b}(\underline{x}) = \sum_{\mathbf{k}} \omega_{\mathbf{k}}\, \hat{b}^\dagger_{\mathbf{k}}\, \hat{b}_{\mathbf{k}}. \qquad (4.7)$$

Poincaré group symmetry. A representation of the full Poincaré group exists on this Hilbert space. The starting point is the generators of the classical theory, suitably normal-ordered to correspond to well-defined operators.

[4] In momentum space we write b and $b^\dagger$ from the well known relation $b_{\mathbf{k}} = \sqrt{\omega_{\mathbf{k}}/2}\phi_{\mathbf{k}} + i/\sqrt{2\omega_{\mathbf{k}}}\pi_{\mathbf{k}}$.

Substituting the fields in terms of creation and annihilation operators, the generators can be written as

$$\hat{P}^i = i\int d^3\underline{x}\; \hat{b}^\dagger(\underline{x})\,\partial^i\,\hat{b}(\underline{x}) \tag{4.8}$$

$$\hat{J}^i = i\epsilon^{ijk}\int d^3\underline{x}\;\; \hat{b}^\dagger(\underline{x})\,x_j\,\partial_k\,\hat{b}(\underline{x}) \tag{4.9}$$

$$\hat{K}^i = \int d^3\underline{x}\; \hat{b}^\dagger(\underline{x})\,{}^n\Gamma^{1/4}\,x^i\,{}^n\Gamma^{1/4}\,\hat{b}(\underline{x}) \tag{4.10}$$

These generators, together with $\hat{H}$ defined in Eq. (4.7), satisfy the Lie algebra relations of the Poincaré group.

In the canonical picture, the covariant fields are obtained by the Heisenberg equations of motion

$$\hat{\phi}(\underline{x},s) := e^{\frac{i}{\hbar}s\hat{H}}\hat{\phi}(\underline{x})e^{-\frac{i}{\hbar}s\hat{H}} = \cos({}^n\Gamma^{\frac{1}{2}}s)\hat{\phi}(\underline{x}) + {}^n\Gamma^{\frac{-1}{2}}\sin({}^n\Gamma^{\frac{1}{2}}s)\hat{\pi}(\underline{x}) \tag{4.11}$$

$$\hat{\pi}(\underline{x},s) := e^{\frac{i}{\hbar}s\hat{H}}\hat{\pi}(\underline{x})e^{-\frac{i}{\hbar}s\hat{H}} = -{}^n\Gamma^{\frac{1}{2}}\sin({}^n\Gamma^{\frac{1}{2}}s)\hat{\phi}(\underline{x}) + \cos({}^n\Gamma^{\frac{1}{2}}s)\hat{\pi}(\underline{x}) \tag{4.12}$$

The explicit automorphisms generated by the boosts may easily be calculated for the Heisenberg picture creation and annihilation operators

$$\hat{b}(\underline{x},s) := e^{\frac{i}{\hbar}s\hat{H}}\;\hat{b}(\underline{x})\;e^{-\frac{i}{\hbar}s\hat{H}} = e^{-is\,{}^n\Gamma}\hat{b}(\underline{x}) \tag{4.13}$$

and they give

$$e^{im_i\hat{K}^i}\;\hat{b}(\underline{x},s)\;e^{-im_i\hat{K}^i} = \hat{b}(\underline{x}',s'), \tag{4.14}$$

where the transformation $(\underline{x},s)\mapsto(\underline{x}',s')$ is given by Eq. (3.27), so that we can write

$$e^{im_i\hat{K}^i}\;\hat{b}(\underline{x},s)\;e^{-im_i\hat{K}^i} = \hat{b}(\Lambda(\underline{x},s)). \tag{4.15}$$

From this, one can write the explicit transformation laws for the Heisenberg fields $\hat{\phi}(\underline{x},s)$ and $\hat{\pi}(\underline{x},s)$.

Some questions that arise in the canonical treatment. The first question that arises in this standard treatment is whether the Poincaré transformations are associated with any changes of foliation. Working canonically *there is no trace of the foliation vector* on the Fock space defined by Eq. (4.6), so this question cannot readily be answered.

Being able to talk about foliations is a necessary step if we are to elucidate the spacetime character of a quantum theory, in which the parameter s of the Heisenberg picture objects corresponds to the foliation time parameter in spacetime. For example, the physical meaning of the parameter s of the Heisenberg objects depends on the choice of foliation vector.

4.2 Histories Quantum Field Theory

Quantum mechanics histories. As we have already mentioned in section 2, the introduction of the history group [2] as an analogue of the canonical group relates the spectral projectors of the generators of its Lie algebra with propositions about history phase space quantities. This algebra is infinite-dimensional and therefore there exist infinitely many representations. However the physically appropriate representation of the smeared history algebra can be uniquely selected by the requirement that the time-averaged energy exists as a proper self-adjoint operator [2]. The resulting Hilbert space has a natural interpretation as a continuous-tensor product: hence by this means we also gain a natural mathematical implementation of the concept of 'continuous' temporal logic.

4.2.1 Histories Hamiltonian algebra

We shall now apply the histories ideas to relativistic quantum field theory on Minkowski space-time. The representation of the history algebra is to be selected by requiring that the time-averaged energy $H_\chi = \int d^4X\, \chi(t)\, H_t$, (which is associated with history propositions about temporal averages of the energy) exists as a proper essentially self-adjoint operator [2]. In what follows, for the sake of typographical simplicity we will no longer use hats to indicate quantum operators.

We start with the abstract algebra

$$[\ \phi(X)\,,\phi(X')\ ] = 0 \tag{4.16}$$

$$[\ \pi(X)\,,\pi(X')\ ] = 0 \tag{4.17}$$

$$[\ \phi(X)\,,\pi(X')\ ] = i\hbar\delta^4(X-X') \tag{4.18}$$

where X and X' are spacetime points.

In order to find suitable representations of this algebra we start with the Fock space $\mathcal{F} := \exp L^2(\mathrm{I\!R}^4, d^4X)$ in which there is a natural definition of

creation and annihilation operators $b(X)$ and $b^\dagger(X)$ that satisfy the commutation relations

$$[\,b(X)\,,b(X')\,]=0 \tag{4.19}$$

$$[\,b^\dagger(X)\,,b^\dagger(X')\,]=0 \tag{4.20}$$

$$[\,b(X)\,,b^\dagger(X')]=\hbar\delta^4(X-X'). \tag{4.21}$$

An appropriate representation of the Poincaré group can be defined by requiring

$$U(\Lambda)\,b(X)\,U(\Lambda)^\dagger \;=\; b(\Lambda X) \tag{4.22}$$

$$U(\Lambda)\,|0\rangle \;=\; |0\rangle \tag{4.23}$$

where $|\,0\rangle$ is the cyclic 'vacuum' state for the theory. Then clearly history fields can be defined by

$$\phi(X) \;:=\; \frac{1}{\sqrt{2}}\Big(b(X)+b^\dagger(X)\Big) \tag{4.24}$$

$$\pi(X) \;:=\; \frac{1}{i\sqrt{2}}\Big(b(X)-b^\dagger(X)\Big). \tag{4.25}$$

and satisfy Eqs. (4.16–4.18). They also transform in the obvious covariant way under the operators $U(\Lambda)$ introduced above.

It should be emphasized that the fields $\phi(X)$ and $\pi(X)$ thus defined *do not* have any foliation vector dependence. However, an operator H_χ of the time-averaged energy of the system *cannot* be well defined so that it depends functionally on these fields in the usual way.

Hence we must seek a different, and more physically appropriate representation, for the history algebra on the history Hilbert space $\mathcal{F}$.

We start by making a *fixed* choice of a unit time-like vector n which we use to foliate the four-dimensional Minkowski space-time. It is clear that the average-energy operator is itself dependent upon the choice of foliation n, and therefore this must also be true for the elements of the history algebra. Hence to emphasise that the physically appropriate representation depends on n we rewrite the history commutation relations as

$$[\;{}^n\phi(X)\,,{}^n\phi(X')\;] \;=\; 0 \tag{4.26}$$

$$[\;{}^n\pi(X)\,,{}^n\pi(X')\;] \;=\; 0 \tag{4.27}$$

$$[\;{}^n\phi(X)\,,{}^n\pi(X')\;] \;=\; i\hbar\delta^4(X-X') \tag{4.28}$$

where X and X' are spacetime points. The dependence of the representation of the history algebra on the choice of the time-like foliation vector n is indicated by the upper left symbol for the field ${}^n\phi(X)$ and its 'conjugate' ${}^n\pi(X)$.

One may also write the canonical version of the history algebra. Notice that—as in the discussion above of classical history theory—in relating Eqs. (4.26)–(4.28) with the canonical version of the history algebra the three-vector $\underline{x}$ may be equated with a four-vector x_n that satisfies $n \cdot x_n = 0$ (the dot product is taken with respect to the Minkowski metric $\eta_{\mu\nu}$) so that the pair $(t, \underline{x}) \in \mathrm{I\!R} \times \mathrm{I\!R}^3$ is associated with the spacetime point $X = tn + x_n$ (in particular, $t = n \cdot X$). The canonical history commutation relations can be written therefore as

$$\begin{aligned}
[\,{}^n\phi(t\,,\underline{x}),{}^n\phi(t'\,,\underline{x}')\,] &= 0 && (4.29)\\
[\,{}^n\pi(t\,,\underline{x}),{}^n\pi(t'\,,\underline{x}')\,] &= 0 && (4.30)\\
[\,{}^n\phi(t\,,\underline{x}),{}^n\pi(t'\,,\underline{x}')\,] &= i\hbar\delta(t-t')\delta^3(\underline{x}-\underline{x}'), && (4.31)
\end{aligned}$$

where, for each $t \in \mathrm{I\!R}$, the fields ${}^n\phi(t\,,\underline{x})$ and ${}^n\pi(t\,,\underline{x})$ are associated with the spacelike hypersurface $\Sigma_t = (n, t)$, characterised by the normal vector n and by the foliation parameter t. In particular, the three-vector $\underline{x}$ in ${}^n\phi(t\,,\underline{x})$ or in ${}^n\pi(t\,,\underline{x})$ denotes a vector in this space.

A central feature of the approach that is followed in this work for the histories quantum field theory, is that for all foliation vectors n, the corresponding foliation-dependent representations of the history algebra Eqs. (4.26)–(4.28) can all be realised on the *same* Fock space $\mathcal{F} = \exp L^2(\mathrm{I\!R}^4, d^4X)$ that also carries the 'covariant' fields $\phi(X)$ and $\pi(X)$ defined in Eqs. (4.24)–(4.25).

The foliation-dependent fields ${}^n\phi(X)$ and ${}^n\pi(X)$ are expressed in terms of the covariant creation and annihilation operators of $\exp L^2(\mathrm{I\!R}^4, d^4X)$, and the related covariant fields $\phi(X)$ and $\pi(X)$ of Eqs. (4.26–4.28), as

$$\begin{aligned}
{}^n\phi(X) &= -\frac{1}{\sqrt{2}}{}^n\Gamma^{1/4}\Big(b(X) + b^\dagger(X)\Big) = -{}^n\Gamma^{1/4}\,\phi(X) && (4.32)\\
{}^n\pi(X) &= \frac{1}{i\sqrt{2}}{}^n\Gamma^{1/4}\Big(b(X) - b^\dagger(X)\Big) = {}^n\Gamma^{1/4}\,\pi(X), && (4.33)
\end{aligned}$$

and conversely,

$$b(X) = \frac{1}{\sqrt{2}}\Big(\phi(X)+i\pi(X)\Big) = \frac{1}{\sqrt{2}}\Big({}^n\Gamma^{1/4}\,{}^n\phi(X)+i\,{}^n\Gamma^{-1/4}\,{}^n\pi(X)\Big) \quad (4.34)$$

$$b^\dagger(X) = \frac{1}{\sqrt{2}}\Big(\phi(X)-i\pi(X)\Big) = \frac{1}{\sqrt{2}}\Big({}^n\Gamma^{1/4}\,{}^n\phi(X)-i\,{}^n\Gamma^{-1/4}\,{}^n\pi(X)\Big) \quad (4.35)$$

where ${}^n\Gamma$ denotes the partial differential operator defined in Eq. (3.17) on the Hilbert space $L^2(\mathbb{R}^4, d^4X)$.

For a fixed foliation vector n, we seek a family of 'internal' Hamiltonians ${}^nH_t$, $t\in\mathbb{R}$, whose explicit formal expression may be deduced from the standard quantum field theory expression to be

$${}^nH_t := \frac{1}{2}\int d^4X\,\Big\{{}^n\pi(X)^2+(n^\mu n^\nu-\eta^{\mu\nu})\,\partial_\mu\,{}^n\phi(X)\,\partial_\nu\,{}^n\phi(X)+\tilde{m}^2\,{}^n\phi(X)^2\Big\}\,\delta(t-n\cdot X) \quad (4.36)$$

The corresponding smeared expression (which must be normal-ordered to be well-defined) is

$$\begin{aligned} {}^nH_\chi &:= \int_{-\infty}^{\infty} dt\,\chi(t)\,{}^nH_t \quad (4.37)\\ &= \frac{1}{2}:\int d^4X\,\Big\{{}^n\pi(X)^2+(n^\mu n^\nu-\eta^{\mu\nu})\,\partial_\mu\,{}^n\phi(X)\,\partial_\nu\,{}^n\phi(X)+\tilde{m}^2\,{}^n\phi(X)^2\Big\}\,\chi(n\cdot X): \end{aligned}$$

where χ is a real-valued test function.

We next augment the history algebra with the following commutation relations that would be satisfied by the operators ${}^nH(\chi)$, if they existed,

$$[\,{}^nH_\chi,\,{}^n\phi(X)\,] = -i\hbar\chi(n\cdot X)\,{}^n\pi(X) \quad (4.38)$$

$$[\,{}^nH_\chi,\,{}^n\pi(X)\,] = i\hbar\chi(n\cdot X)\,{}^n\Gamma\,{}^n\phi(X) \quad (4.39)$$

$$[\,{}^nH_\chi,\,{}^nH_{\chi'}\,] = 0. \quad (4.40)$$

If the operators ${}^nH$ existed, the above commutation relations would give rise to the transformations

$$\begin{aligned} e^{\frac{i}{\hbar}{}^nH_\chi}\,{}^n\phi(X)\,e^{\frac{-i}{\hbar}{}^nH_\chi} &= \quad (4.41)\\ &= \cos\Big[\chi(n\cdot X){}^n\Gamma^{\frac{1}{2}}\Big]\,{}^n\phi(X)+{}^n\Gamma^{\frac{-1}{2}}\sin\Big[\chi(n\cdot X){}^n\Gamma^{\frac{1}{2}}\Big]\,{}^n\pi(X) \end{aligned}$$

$$\begin{aligned} e^{\frac{i}{\hbar}{}^nH_\chi}\,{}^n\pi(X)\,e^{\frac{-i}{\hbar}{}^nH_\chi} &= \quad (4.42)\\ &= -{}^n\Gamma^{\frac{1}{2}}\sin\Big[\chi(n\cdot X){}^n\Gamma^{\frac{1}{2}}\Big]\,{}^n\phi(X)+\cos\Big[\chi(n\cdot X){}^n\Gamma^{\frac{1}{2}}\Big]\,{}^n\pi(X) \end{aligned}$$

Note that the expression $\chi(n\cdot X)^n\Gamma^{\frac{1}{2}}$ is unambiguous since, viewed as an operator on $L^2(\mathbb{R}^4, d^4X)$, multiplication by $\chi(n\cdot X)$ commutes with ${}^n\Gamma^{\frac{1}{2}}$.

The right hand side of Eqs. (4.41)–(4.42) defines an automorphism of the history algebra Eqs. (4.26)–(4.28), and all that remains is to show that these automorphisms are unitarily implementable in this representation. To this end, we use Eqs. (4.34)–(4.35) to prove that

$$e^{i\,{}^nH_\chi/\hbar}\, b(X)\, e^{-i\,{}^nH_\chi/\hbar} = e^{-i\,\chi(n\cdot X)\;{}^n\Gamma^{\frac{1}{2}}}\, b(X). \tag{4.43}$$

However, the operator defined on $L^2(\mathbb{R}^4, d^X)$ by

$$(O(\chi)\psi)(X) := e^{-i\chi(n\cdot X)\;{}^n\Gamma^{\frac{1}{2}}}\,\psi(X) \tag{4.44}$$

is easily seen to be unitary, and hence we conclude [2] that the desired quantities ${}^nH_\chi$ exist as self-adjoint operators on the Fock space $\mathcal{F}$ associated with the creation and annihilation operators $b^\dagger(X)$ and $b(X)$. The spectral projectors of these operators ${}^nH_\chi$ represent propositions about the time-averaged value of the energy in the spacetime foliation determined by n.

To conclude: for each *fixed* choice of a foliation vector n, we have a physically meaningful representation of the history algebra Eqs. (4.26)–(4.28) on the Hilbert space $\mathcal{F} = \exp L^2(\mathbb{R}^4, d^4X)$. Thus the same Hilbert space $\mathcal{F}$ carries *all* different representations—for different choices of n—of the quantum field theory history algebra.

4.2.2 The action operator

We now define the action ${}^nS_\chi$ and the Liouville ${}^nV$ operators as normal-ordered versions of their classical analogues

$$\begin{aligned}
{}^nS_\chi &= {}^nV - \frac{1}{2} : \int d^4X\; \{{}^n\pi^2(X) + {}^n\phi(X)\;{}^n\Gamma\;{}^n\phi(X)\}\chi(n\cdot X) : &(4.45)\\
{}^nV &= : \int_{-\infty}^{\infty} d^4X\; {}^n\pi(X)\, n^\mu\partial_\mu\, {}^n\phi(X) : &(4.46)
\end{aligned}$$

The automorphisms of the history algebra that are generated by the action and Liouville operators are

$$\begin{aligned}
e^{is\,{}^nS_\chi/\hbar}\, b(X)\, e^{-is\,{}^nS_\chi/\hbar} &= e^{-i\int_{s'-s}^{s'+s} ds'\,\chi(nX+s')\,{}^n\Gamma^{\frac{1}{2}} - sn^\mu\partial_\mu}\, b(X) &(4.47)\\
e^{is\,{}^nV/\hbar}\, b(X)\, e^{-is\,{}^nV/\hbar} &= e^{-s\,n^\mu\partial_\mu}\, b(X), &(4.48)
\end{aligned}$$

and are easily shown to be unitarily implementable. In what follows, the real-valued smearing function χ is set equal to $\chi(t) = 1$ for every $t \in \mathbb{R}$.

4.3 Poincaré group covariance

A significant feature of the histories formalism is the temporal structure of the theory. It introduces a new approach to the concept of time, in which time is distinguished as an ordering parameter (logical structure), and as an evolution parameter (dynamics). In particular, as we have already shown in non-relativistic quantum mechanics [1], the Liouville operator ${}^{n}V$ generates time translations with respect to the 'external' t-time parameter, and the Hamiltonian operator ${}^{n}H$ generates time translations with respect to the 'internal' evolution s-time parameter. The action operator ${}^{n}S$ generates *both* types of time transformations; it is the time generator for the histories theory for solutions of the equations of motion [5]. The same construction is true for a histories quantum field theory.

The invariance of standard quantum field theory under the Poincaré group, has been a difficult issue to address for many years. In a canonical treatment of quantum field theory, the Schrödinger-picture fields depend on the reference frame (*i.e.*, choice of foliation). In order to demonstrate manifest independence of this choice with the aid of Heisenberg-picture fields, one still has to contend with the foliation-dependence of the Hamiltonian that generates the Heisenberg fields.

In histories theory, the enhanced temporal structure enables the study of a Poincaré group transformation between different foliations. In particular we will show that different representations corresponding to different foliation vectors n, are related by Lorentz boosts of the 'external' Poincaré group:

$$U(\Lambda)\, {}^{n}\!\phi(X)\, U(\Lambda)^{-1} = {}^{\Lambda n}\!\phi(\Lambda X) \tag{4.49}$$

and where the time translations generator is closely related to the 'Liouville' operator V.

[5] In histories theory the physical time translation generator is the action operator ${}^{n}S$; both Liouville ${}^{n}V$ and Hamiltonian ${}^{n}H$ operators are time translation generators that correspond to two different aspects (two modes) of the notion of time. However, only ${}^{n}S$ is related to the actual physical time parameter, in analogy with the standard theory where the Hamiltonian ${}^{n}H$ is the time translation generator.

The Heisenberg-picture operators. We first define the Heisenberg-picture analogue of the scalar field, to illustrate the different time translations associated with the two time labels. We use a similar notation to that in the classical case: *i.e.*, the Heisenberg-picture field is written as ${}^n\phi(X,s) = {}^n\phi(t,\underline{x},s)$, where the space-time point $X=(t,\underline{x})$ is expressed in coordinates adapted to n. Thus

$$\begin{aligned}
{}^n\phi(X,s) &= {}^n\phi(t,\underline{x},s) := e^{\frac{i}{\hbar}s\,{}^nH}\,{}^n\phi(t,\underline{x})\,e^{-\frac{i}{\hbar}s\,{}^nH} \\
&= \cos\left(s\,{}^n\Gamma^{\frac{1}{2}}\right){}^n\phi(X) + {}^n\Gamma^{\frac{-1}{2}}\sin\left(s\,{}^n\Gamma^{\frac{1}{2}}\right){}^n\pi(X) \qquad (4.50)\\
{}^n\pi(X,s) &= {}^n\pi(t,\underline{x},s) := e^{\frac{i}{\hbar}s\,{}^nH}\,{}^n\pi(t,\underline{x})\,e^{-\frac{i}{\hbar}s\,{}^nH} \\
&= -{}^n\Gamma^{\frac{1}{2}}\sin\left(s\,{}^n\Gamma^{\frac{1}{2}}\right){}^n\phi(X) + \cos\left(s\,{}^n\Gamma^{\frac{1}{2}}\right){}^n\pi(X). \qquad (4.51)
\end{aligned}$$

The different types of time translation are particularly easy to see by studying the action of the Liouville ${}^nV$ and action ${}^nS$ operators on the Heisenberg-picture fields $b(X,s)$

$$e^{i\tau{}^nH}\,b(t,\underline{x},s)\,e^{-i\tau{}^nH} := b(t\,,\underline{x}\,,s+\tau) \qquad (4.52)$$

$$e^{i\tau{}^nV}\,b(t,\underline{x},s)\,e^{-i\tau{}^nV} := b(t+\tau\,,\underline{x}\,,s) \qquad (4.53)$$

$$e^{i\tau{}^nS}\,b(t,\underline{x},s)\,e^{-i\tau{}^nS} := b(t+\tau\,,\underline{x}\,,s+\tau) \qquad (4.54)$$

The label s corresponds to the 'internal' time of the unitary Hamiltonian time evolution, while t corresponds to the 'external' time that labels the time-ordering of events in a history for the Shrödinger-picture operators.

4.3.1 The internal Poincaré group

As we showed previously, each fixed choice of foliation vector n corresponds to a *different* representation of the history algebra on the *same* Fock space $\mathcal{F} = \exp L^2(\mathbb{R}^4, d^4X)$. Hence, we may heuristically say[6], that, for a given vector n, and for each value of the associated time t, there will be a Hilbert space $\mathcal{H}_t$ that carries an independent copy of the standard quantum field theory. In particular, there exists a representation of the Poincaré group associated with each spacelike slice $(n\,,t)$, where $t \in \mathbb{R}$.

[6] The physical quantities in histories appear naturally space-time averaged, therefore they are smeared with appropriate test functions. Strictly speaking, quantities labeled at moments of time are not well-defined mathematically.

In what follows, a particularly important role will be assigned to the averaged 'internal' Poincaré group. For example, we define the averaged energy ${}^nH := \int d^4X\, {}^nH_t$ that generates translations on the s-time parameter of the Heisenberg-picture fields ${}^n\phi(X,s) = {}^n\phi(t\,,\underline{x}\,,s)$, without affecting the 'external' t-time parameter:

$$ {}^n\phi(X,s) = \xrightarrow{{}^nH} {}^n\phi(X,s+s') \tag{4.55}$$

$$ {}^n\pi(X,s) = \xrightarrow{{}^nH} {}^n\pi(X,s+s'). \tag{4.56}$$

The expressions for the 'internal' Poincaré generators of spatial translations P^i, and rotations J^i can be written in direct analogy with the expressions Eqs. (3.21)–(3.23) of the classical case. We use the normal-ordered expressions

$$\begin{aligned} P(m) &= \; : \int d^4X\, \pi(X)\, m^\mu \partial_\mu\, \phi(X) : \\ &= i \int d^4X\, b^\dagger(X)\, m^\mu \partial_\mu\, b(X) \end{aligned} \tag{4.57}$$

$$\begin{aligned} J(m) &= \frac{1}{2} n_\mu m_\nu \epsilon^{\mu\nu\rho\sigma} : \int d^4X\, \pi(X)\, X_\rho\, \partial_\sigma\, \phi(X) : \\ &= i\frac{1}{2} n_\mu m_\nu \epsilon^{\mu\nu\rho\sigma} \int d^4X\, b^\dagger(X)\, X_\rho\, \partial_\sigma\, b(X) \end{aligned} \tag{4.58}$$

We have used an 'pseudo-covariant' notation by employing a n-spacelike vector m (*i.e.,* such that $(n_\mu m^\mu = 0)$. Note that the terms involving a pair of creation operators, or a pair of annihilation operators, can be shown to vanish through integration by parts.

Of particular interest, is the action of the boost generator ${}^nK(m)$ defined as

$$ {}^nK(m) = \; m_\mu : \int d^4X\, [n\cdot X\, {}^n\pi\, \partial^\mu\, {}^n\phi - \frac{1}{2} X^\mu\, ({}^n\pi^2 + {}^n\phi\, {}^n\Gamma\, {}^n\phi)] : \tag{4.59}$$

$$ = \int d^4X\, b^\dagger(X)\, {}^n\Gamma^{\frac{1}{4}}\, X^\mu m_\mu\, {}^n\Gamma^{\frac{1}{4}}\, b(X). \tag{4.60}$$

The key feature of the boost generator ${}^nK(m)$ is that it mixes the s-time parameter with the three-vectors $\underline{\mathrm{x}}$. The action of these boost transformations is most clearly seen on the Heisenberg objects $\phi(X,s) = \phi(t,\underline{x},s)$

$$ {}^{\mathrm{int}}U(\Lambda)\, {}^n\phi(t\,,\underline{x}\,,s)\, {}^{\mathrm{int}}U(\Lambda)^{-1} = {}^n\phi(t\,,\Lambda(\underline{x}\,,s)), \tag{4.61}$$

where ${}^{\rm int}U(\Lambda) := e^{iK(m)}$ is the unitary operator that generates Lorentz transformations, and Λ is the Lorentz transformation generated by m.

At this point we note the action of the internal Poincaré group on the action ${}^{n}\!S$, Hamiltonian ${}^{n}\!H$ and Liouville ${}^{n}V$ operators respectively:

$$ {}^{\rm int}U(\Lambda)\ {}^{n}\!H\ {}^{\rm int}U(\Lambda)^{-1} = {}^{n}\!H \tag{4.62} $$
$$ {}^{\rm int}U(\Lambda)\ {}^{n}V\ {}^{\rm int}U(\Lambda)^{-1} = {}^{n}V \tag{4.63} $$
$$ {}^{\rm int}U(\Lambda)\ {}^{n}\!S\ {}^{\rm int}U(\Lambda)^{-1} = {}^{n}\!S. \tag{4.64} $$

As we would expect from standard canonical quantum field theory, we see that the above operators remain invariant under the 'internal' Lorentz transformations.

4.3.2 External Poincaré group

A key result in histories classical field theory is that there also exists a second—the 'external'—Poincaré group symmetry of the theory, with generators

$$ \tilde{P}^{\mu} \;=\; :\int d^4X\, \pi(X)\, \partial^{\mu}\, \phi(X) : \tag{4.65} $$
$$ \tilde{M}^{\mu\nu} \;=\; :\int d^4X\, \pi(X)\, (X^{\mu}\partial^{\nu} - X^{\nu}\partial^{\mu})\, \phi(X) : \tag{4.66} $$

Note that these definitions use the covariant fields $\phi(X)$ and $\pi(X)$ that satisfy the algebra Eqs. (4.16)–(4.18) rather than the foliation-dependent fields ${}^{n}\phi(X)$ and ${}^{n}\pi(X)$ of Eqs. (4.26)–(4.28). However, many of the generators of the external Poincaré group are exactly the same whether one uses covariant fields expressions or foliation-dependent ones: they differ only for the case of the boosts generators ${}^{n}K(m)$.

In particular, the Liouville operator $\tilde{P}^0 = V$, given by the expression

$$ \begin{aligned} V \;&=\; :\int d^4X\, \pi(X) n^{\mu}\partial_{\mu}\, \phi(X) : \\ &=\; i\int d^4X\, b^{\dagger}(X) n^{\mu}\partial_{\mu} b(X) \end{aligned} \tag{4.67} $$

generates translations on the time label t.

The space translations and rotation generators are identical to those of the internal Poincaré group Eqs. (4.57–4.58). However the external boost

generator $\tilde{K}(m)$ differs from the internal one ${}^nK(m)$, and hence it is of particular interest to study the action of the former.

The generator of time-translations V acts on Schrödinger picture objects as

$$ {}^n\phi(X) = {}^n\phi(t\,,\underline{x}) \;\overset{V}{\longrightarrow}\; {}^n\phi(t+\tau\,,\underline{x}) \tag{4.68} $$

$$ {}^n\pi(X) = {}^n\pi(t\,,\underline{x}) \;\overset{V}{\longrightarrow}\; {}^n\pi(t+\tau\,,\underline{x}). \tag{4.69} $$

The 'external' boost generator $\tilde{K}(m)$ is

$$ \begin{aligned} \tilde{K}(m) &= \,:\int_{-\infty}^{\infty} d^4X\,\pi(X)\,T_m\,\phi(X)\,: && (4.70)\\ &= i\int_{-\infty}^{\infty} d^4X\,b^\dagger(X)\,T_m\,b(X) && (4.71) \end{aligned} $$

where we define the operator T_m as

$$ (T_m f)(X) := n_\mu m_\nu (X^\mu \partial^\nu - X^\nu \partial^\mu) f(X). \tag{4.72} $$

and $n \cdot m = 0$. Then the boost generator $\tilde{K}(m)$ acts on the fields ${}^n\phi(X)$ as

$$ {}^{\mathrm{ext}}U(\Lambda)\;{}^n\phi(X)\;{}^{\mathrm{ext}}U(\Lambda)^{-1} = \;{}^{\Lambda n}\phi(\Lambda(X)), \tag{4.73} $$

and it mixes the t-time parameter with the three-vector $\underline{\mathrm{x}}$. However, the crucial point is that $\tilde{K}(m)$ generates Lorentz transformations *on the foliation vector* n as well.

This can be viewed as a demonstration of explicit Poincaré covariance, as we can see from the action of the external Lorentz transformations on the Heisenberg-picture fields ${}^n\phi(X\,,s)$ as

$$ {}^{\mathrm{ext}}U(\Lambda)\;{}^n\phi(X,s)\;{}^{\mathrm{ext}}U(\Lambda)^{-1} = \;{}^{\Lambda n}\phi(\Lambda(X,s)). \tag{4.74} $$

The generators of the *internal* Poincaré group transform under the action of the *external* Poincaré group as

$$ \begin{aligned} {}^{\mathrm{ext}}U(\Lambda)\;{}^nH\;{}^{\mathrm{ext}}U(\Lambda)^{-1} &= \;{}^{\Lambda n}H && (4.75)\\ {}^{\mathrm{ext}}U(\Lambda)\;{}^n\tilde{K}(m)\;{}^{\mathrm{ext}}U(\Lambda)^{-1} &= \;{}^{\Lambda n}\tilde{K}(\Lambda m). && (4.76) \end{aligned} $$

Of considerable importance is the fact that the action operator ${}^{n}S$ transforms in the same way:

$$^{\rm ext}U(\Lambda)\ {}^{n}S\ {}^{\rm ext}U(\Lambda)^{-1} = {}^{\Lambda n}S \tag{4.77}$$

Hence the action of the external Poincaré group *relates* representations of the theory that *differ* with respect to the foliation vector n. As we shall see in the following section, this is crucial when we discuss the Poincaré invariance of probabilities.

In summary, we have showed that the history version of quantum field theory carries representations of two Poincaré groups. The 'internal' Poincaré group is defined in analogy to the one in the standard canonical treatment of the theory. It corresponds to time-translations with respect to the 'internal' s-time parameter of histories theory. The Lorentz part of the 'external' Poincaré group intertwines representations of the theory associated with different choices of foliation, all of which however are realised on the *same* Fock space $\mathcal{F}$. It corresponds to time-translations with respect to the 'external' t-time parameter.

The translation parts of these two types of Poincaré transformation—corresponding to the relations between the t time parameter and kinematics, and the s time parameter and dynamics—have very significant analogues in the case of the histories version of general relativity [9].

4.3.3 The decoherence functional

'Classical' coherent states. In [1], we showed how a classical-quantum relation can be nicely described in histories theory by using the history analogue of coherent states. In the histories formalism, a non-normalised coherent state vector is written as [2]

$$|\exp z\rangle = \oplus_{n=0}^{\infty}(n!)^{-\frac{1}{2}}(\otimes|z\rangle)^{n}. \tag{4.78}$$

The corresponding normalised coherent states can be obtained by unitary transformations of the vacuum state as

$$|z\rangle := \frac{1}{\sqrt{\langle \exp z|\exp z\rangle}}|\exp z\rangle = U[f,h]|0\rangle \tag{4.79}$$

where $U[f,h]$ is the Weyl operator defined as

$$U[f,h] := e^{\frac{i}{\hbar}(\ {}^n\phi(f)\ -\ {}^n\pi(h)\ )}, \tag{4.80}$$

and f and h are smearing functions that belong to $L^2(\mathbb{R}^4, d^4X)$. We write the normalised coherent state $|z\rangle$ corresponding to the pair f, h as $|f,h\rangle$. In this context we know that f and h correspond to classical values and therefore correspond to a path on classical phase space. In this correspondence, the functions f and h are the classical values of the field $\phi(X)$ and its conjugate momenta $\pi(X)$, respectively.

The set of all coherent states is independent of the choice of foliation since these coherent states are eigenstates of the annihilation operator $b(X)$, which is foliation independent. However, the physical identification of the vector $|z\rangle$ with a phase space path *is* foliation-dependent since it depends on the Weyl operator, which itself depends on the choice of the representation of the history algebra on the Fock space $\mathcal{F}$[7]. One should recall that the space of classical histories Π is itself dependent on the choice of foliation.

So far our discussion of the histories version of quantum field theory has been at the level of field algebras and group transformations. However, in histories formalism physically crucial 'probabilistic' information is contained in the decoherence functional.

In this HPO formalism, the most general form for the decoherence functional of a pair of history propositions α, β is

$$d(\alpha,\beta) = Tr_{\mathcal{F}\times\mathcal{F}}\left(\alpha\otimes\beta\,\Xi\right), \tag{4.81}$$

in terms of an operator Ξ on $\mathcal{F}\times\mathcal{F}$ [10].

In our case, the operator Ξ reads

$$\Xi := \langle 0|\rho_{-\infty}|0\rangle(\mathcal{S}_{cts}\mathcal{U})^\dagger \otimes (\mathcal{S}_{cts}\mathcal{U}), \tag{4.82}$$

in terms of the operator $\mathcal{S}_{cts}\mathcal{U}$ that we proved in [1] that it is an implicit function of the action operator: therefore there is an implicit dependence of Ξ on the foliation vector n. The matrix elements of $\mathcal{S}_{cts}\mathcal{U}$ in a coherent state basis can written in terms of the classical action functional $S[f,h]$ as

$$\langle f,h|\mathcal{S}_{cts}\mathcal{U}|f,h\rangle = e^{iS[f,h]}. \tag{4.83}$$

[7]Given a complex path z, the classical phase space path (f,h) is defined by the foliation-dependent expression $z = {}^n\Gamma^{1/4} f + i\ {}^n\Gamma^{-1/4} h$

The explicit relation of $\mathcal{S}_{cts}\mathcal{U}$ with the action operator ${}^nS$ is as follows. For a general operator A on $L^2(\mathbb{R}^4, d^4X)$ one can define an operator $\Gamma(A)$ on $\mathcal{F}$ as

$$\Gamma(A)|\exp z\rangle = |Az\rangle. \tag{4.84}$$

In our case we have

$$e^{is^nS} = \Gamma(e^{is^n\sigma}) \tag{4.85}$$

$$\mathcal{S}_{cts}\mathcal{U} = \Gamma(1 + i^n\sigma), \tag{4.86}$$

in terms of the operator ${}^n\sigma = n^\mu\partial_\mu - {}^n\Gamma^{1/2}$. Hence, the decoherence functional depends on the representation through the phase space action ${}^nS$.

This raises the critical issue of the physical meaning of the fact that the formalism appears to depend on a specific choice of the foliation vector n. We have seen above that the representation of the phase space quantities by Hilbert space operators depends on n, and that there exist unitary intertwiners between different representations given by the boosts of the external Poincaré group. As has been discussed in [11], a transformation law for the observables by means of a unitary operator U

$$\alpha \to \alpha' = U\alpha U^\dagger \tag{4.87}$$

implies that the operator Ξ of the decoherence functional, carrying a label for the foliation dependence n, ought to transform as

$${}^n\Xi \to {}^{\Lambda n}\,\Xi = (U \otimes U)\;{}^n\Xi\;(U^\dagger \otimes U^\dagger) \tag{4.88}$$

so that the values of the decoherence functional (corresponding to probabilities and correlation functions of the theory) *are representation-independent*

$${}^{\Lambda n}d(\,{}^{\Lambda n}\alpha\,, {}^{\Lambda n}\beta\,) = {}^nd({}^n\alpha\,, {}^n\beta), \tag{4.89}$$

where ${}^{\Lambda n}d$ is the decoherence functional defined with reference to the operator ${}^{\Lambda n}\Xi$.

In our case we have $U = e^{i\tilde{K}(m)} = {}^{ext}U(\Lambda)$. This changes the foliation dependence of the fundamental fields ${}^n\phi(X)$ and ${}^n\pi(X)$, and hence of any observable ${}^n\alpha$ that depends upon them

$${}^n\alpha \to {}^{\Lambda n}\alpha := U(\Lambda)\;{}^n\alpha\;U(\Lambda)^\dagger \tag{4.90}$$

Some physically interesting examples of observables, in this sense, are integrals $\int d^X\, {}^n\phi(X) f(X)$ of fields ${}^n\phi(X)$, smeared with appropriate test functions $f(X)$, that satisfy $f(X) = f(\Lambda(X)$; another example is any space-time average of the normal-ordered polynomial functions of these fields.

In order to see, how the boosts generator acts on ${}^n\Xi$, it suffices to check its action on $\mathcal{S}_{cts}\mathcal{U}$. This is

$$U\ {}^n\mathcal{S}_{cts}\mathcal{U}\ U^\dagger = \Gamma(1 + ie^{-T_m}\ {}^n\sigma\ e^{T_m}) = \Gamma(1 + i\ {}^{\Lambda n}\sigma) \tag{4.91}$$

Consequently the operator ${}^n\Xi$ transforms as ${}^n\Xi \rightarrow^{\Lambda n} \Xi$. Hence the values of the decoherence functional are foliation independent

$${}^n d({}^n\alpha\,, {}^n\beta) = {}^{\Lambda n} d({}^{\Lambda n}\alpha\,, {}^{\Lambda n}\beta). \tag{4.92}$$

5 Conclusions

We have studied both the classical and the quantum history versions of scalar field theory. We have showed that, in both cases, the crucial feature of the history field theory is the appearance of two Poincaré groups, in direct analogy to the two types of time transformation that characterizes the history formalism. The internal Poincaré group is related to time as an ordering parameter (the Hamiltonian H is the time translations generator), and it is in analogy to the Poincaré group of standard field theory. On the other hand, the external Poincaré group is related to time as a parameter of evolution (the Liouville V is the time translations generator), and it is of particular interest for the quantum case, as it relates representations of the quantum field theory, for different choices of foliation, with Poincaré transformations.

These results will be proved of great importance in the study of history general relativity theory in [9]. In particular, the histories formalism is suitable to deal with issues that lie at the level of the interplay between quantum theory and the spacetime structure. The present work focuses on quantum field theory in a fixed spacetime, however the techniques involved and the concepts introduced, have been able to precisely identify the relation between the quantum mechanical observables and the necessary notion of the spacetime foliation. Many issues are raised at the level of the meaning of reference frames in quantum theory—a foliation corresponds to a reference frame—and more importantly at the level of quantum gravity.

The latter is eventually the aim of the histories programme, and this involves a further elucidation of the meaning of spacetime in a quantum theory. What strikes us as relevant at present is that, one might have to disentangle between the two different views of spacetime transformations: the *passive* and the *active* view. This is subtly hinted by the fact that the transformations generated by the external Poincaré group should be viewed in the passive sense, since the argument X cannot be identified with a fixed, absolute spacetime point *in all representations*.

In order to successfully address the above issues we must first study the history version of general relativity; this is the context of the forthcoming paper [9].

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References

[1] K. Savvidou. The action operator in continuous time histories. *J. Math. Phys.* 40: 5657, (1999);
K. Savvidou. Continuous Time in Consistent Histories. PhD Thesis in gr-qc/9912076, (1999).

[2] C.J. Isham. *J. Math. Phys.* 35:2157, (1994);
C.J. Isham and N. Linden. *J. Math. Phys.* 35:5452, (1994) ;
C.J.Isham and N. Linden. *J. Math. Phys.* 36: 5392, (1995);
C. Isham, N. Linden, K. Savvidou and S. Schreckenberg. *J. Math. Phys.* 37:2261, 1998.

[3] K.Savvidou and C.Anastopoulos. Histories quantisation of parametrised systems: I. Development of a general algorithm. *Class. Quant. Grav.* 17: 2463,2000. (2000).

[4] C. Anastopoulos. Quantum Fields in Nonstatic background: a Histories Perspective. *J. Math. Phys.* February 2000. gr-qc/ 9903026.

[5] R. B. Griffiths. *J. Stat. Phys.* 36: 219 (1984);
R. Omnès. The Interpretation of Quantum Mechanics. *Princeton University Press, Princeton*, (1994) ;
M. Gell-Mann and J. B. Hartle. In *Complexity, Entropy and the Physics of Information*, edited by W. Zurek. *Addison Wesley, Reading*, (1990) ;
J. B. Hartle. Spacetime quantum mechanics and the quantum mechanics of spacetime. In *Proceedings on the 1992 Les Houches School,Gravitation and Quantisation* (1993).

[6] H. Araki. Hamiltonian formalism and the canonical commutation relations in quantum field theory. *J. Math. Phys.* 1:492, 1960.

[7] P.A.M. Dirac. The Lagrangian in quantum mechanics. In *Selected papers on quantum electrodynamics*, edited by J.Schwinger. Dover Publications, Inc. New York (1958).

[8] C. Itzykson and J. Zuber . Quantum field theory. McGraw-Hill, London, 1980.

[9] K. Savvidou. General Relativity Histories Theory. Work in preparation. February 2001.

[10] C.J. Isham, N. Linden and S. Schreckenberg. The classification of decoherence functionals: an analogue of Gleason's theorem. *J. Math. Phys.* 35:6360–6370, 1994;

[11] S. Schreckenberg. Symmetries of Decoherence Functionals. *J. Math. Phys.* 38: 759-769, 1997.